

AN ALGEBRAIC SOLUTION OF EINSTEIN'S FIELD EQUATIONS IN X_4

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ABSTRACT. The main goal in the present paper is to obtain a particular solution $g_{\lambda\mu}$, $\Gamma_{\lambda\mu}^\nu$ and an algebraic solution $\bar{g}_{\lambda\mu}$, $\bar{\Gamma}_{\lambda\mu}^\nu$ by means of $g_{\lambda\mu}$, $\Gamma_{\lambda\mu}^\nu$ in UFT X_4 .

1. Introduction

Einstein([1]) proposed a new unified field theory that would include both gravitation and electromagnetism. Hlavatý([6]) gave the mathematical foundation of the Einstein's unified field theory in a 4-dimensional generalized Riemannian space X_4 (i.e., space-time) for the first time. Since then this theory had been generalized in a generalized Riemannian manifold X_n , the so-called *Einstein's n-dimensional unified field theory*(UFT X_n), and many consequences of this theory has been obtained by a number of mathematicians. However, it has been unable yet to represent a general n-dimensional Einstein's connection in a surveyable tensorial form, probably due to the complexity of the higher dimensions. The purpose of the present paper is to obtain a particular solution $g_{\lambda\mu}$, $\Gamma_{\lambda\mu}^\nu$ of Einstein's field equation in UFT X_4 . In the next, we shall obtain an algebraic solution $\bar{g}_{\lambda\mu}$, $\bar{\Gamma}_{\lambda\mu}^\nu$ by means of $g_{\lambda\mu}$, $\Gamma_{\lambda\mu}^\nu$ in UFT X_4 .

2. Preliminary

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U; x^\nu\}$, where, here and in the sequel, Greek indices run over the range $\{1, 2, \dots, n\}$ and follow

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the summation convention. The algebraic structure on X_n is imposed by a basic real non-symmetric tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$(2.1) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

where we assume that

$$(2.2) \quad (a) \det((g_{\lambda\mu})) < 0, \quad (b) \det((h_{\lambda\mu})) < 0, \quad (c) \det((k_{\lambda\mu})) \geq 0.$$

Since $\det((h_{\lambda\mu})) \neq 0$, we may define a unique tensor $h^{\lambda\nu} (= h^{\nu\lambda})$ by

$$(2.3) \quad h_{\lambda\mu} h^{\lambda\nu} = \delta_{\mu}^{\nu}.$$

We use the tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined on X_n in the usual manner. The manifold X_n is assumed to be connected by a general real connection $\Gamma_{\lambda\mu}^{\nu}$ which may also be split into its symmetric part $\Lambda_{\lambda\mu}^{\nu}$ and skew-symmetric part $S_{\lambda\mu}^{\nu}$, called the *torsion tensor* of $\Gamma_{\lambda\mu}^{\nu}$.

The *Einstein's n-dimensional unified field theory in X_n* (UFT X_n) is governed by the following set of equations :

$$(2.4) \quad \partial_{\omega} g_{\lambda\mu} - g_{\alpha\mu} \Gamma_{\lambda\omega}^{\alpha} - g_{\lambda\alpha} \Gamma_{\omega\mu}^{\alpha} = 0 \quad (\partial_{\nu} = \frac{\partial}{\partial x^{\nu}}),$$

and

$$(2.5) \quad (a) S_{\lambda} = S_{\lambda\alpha}^{\alpha} = 0, \quad (b) R_{[\lambda\mu]} = \partial_{[\lambda} P_{\mu]}, \quad (c) R_{(\lambda\mu)} = 0,$$

where P_{μ} is an arbitrary vector, called the *Einstein's vector*, and $R_{\lambda\mu}$ is the contracted curvature tensor $R_{\lambda\mu\alpha}^{\alpha}$ of the curvature tensor $R_{\lambda\mu\nu}^{\omega}$:

$$(2.6) \quad R_{\lambda\mu\nu}^{\omega} = \partial_{\mu} \Gamma_{\lambda\nu}^{\omega} - \partial_{\nu} \Gamma_{\lambda\mu}^{\omega} + \Gamma_{\lambda\nu}^{\alpha} \Gamma_{\alpha\mu}^{\omega} - \Gamma_{\lambda\mu}^{\alpha} \Gamma_{\alpha\nu}^{\omega}.$$

The equation (2.4) is called the *Einstein's equation*, and the solution $\Gamma_{\lambda\mu}^{\nu}$ of the Einstein's equation is called an *Einstein's connection*. And the vector S_{λ} , defined by (2.5)(a), is called the *torsion vector*.

The following two theorems were proved by Lee([3]).

THEOREM 2.1. *In UFT X_n , if the system (2.4) admits a solution $\Gamma_{\lambda\mu}^{\nu}$ such that its torsion tensor is, for some nonzero vector Y_{λ} ,*

$$(2.7) \quad S_{\lambda\mu}^{\nu} = \frac{2}{n-1} \delta_{[\lambda}^{\nu} k_{\mu]\alpha} Y^{\alpha} + k_{\lambda\mu} Y^{\nu},$$

then it must be of the form

$$(2.8) \quad \Gamma_{\lambda\mu}^{\nu} = \{\lambda^{\nu}{}_{\mu}\} + \frac{2(2-n)}{n-1} k_{(\lambda}{}^{\nu} k_{\mu)\alpha} Y^{\alpha} + \frac{2}{n-1} \delta_{[\lambda}^{\nu} k_{\mu]\alpha} Y^{\alpha} + k_{\lambda\mu} Y^{\nu},$$

where $\{\lambda^{\nu}{}_{\mu}\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$.

THEOREM 2.2. *In UFT X_n , the connection (2.8) is an Einstein's connection if and only if the vector Y_λ defining (2.8) satisfies the following condition*

$$(2.9) \quad \nabla_\nu k_{\lambda\mu} = \frac{2}{n-1} h_{\nu[\lambda} k_{\mu]\alpha} Y^\alpha - 2k_{\nu[\lambda} Y_{\mu]} + \frac{2(n-2)}{n-1} {}^{(2)}k_{\nu[\lambda} k_{\mu]\alpha} Y^\alpha,$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to $\{\lambda^\nu{}_\mu\}$.

3. A particular solution of field equations in UFT X_4

In this section we shall display a particular solution of (2.4) and (2.5) in UFT X_4 . Let a tensor $g_{\lambda\mu}$ be given by the following matrix :

$$(3.1) \quad ((g_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & -e^t & e^t \\ 0 & 1 & 0 & 0 \\ e^t & 0 & 1 & 0 \\ -e^t & 0 & 0 & -1 \end{pmatrix},$$

where $t = x^3 - x^4$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ given by the following matrices :

$$(3.2) \quad ((h_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$(3.3) \quad ((k_{\lambda\mu})) = \begin{pmatrix} 0 & 0 & -e^t & e^t \\ 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 \\ -e^t & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$(3.4) \quad (a) \det((g_{\lambda\mu})) = -1, \quad (b) \det((h_{\lambda\mu})) = -1, \quad (c) \det((k_{\lambda\mu})) = 0,$$

we can choose the tensor $g_{\lambda\mu}$ given by (3.1) as a basic tensor in UFT X_4 , by the assumption (2.2). On the other hand, in virtue of (3.2), all the Christoffel symbols $\{\lambda^\nu{}_\mu\}$ vanish. Hence the components of the first covariant derivatives with respect to $\{\lambda^\nu{}_\mu\}$ are ordinary derivatives, and $H_{\lambda\mu\nu}^\omega = 0$. Define two vectors by

$$(3.5) \quad (a) A_\lambda : (0, 0, 1, -1), \quad (b) B_\lambda : (e^t, 0, 0, 0).$$

Then the skew-symmetric part $k_{\lambda\mu}$ satisfies the following relation :

$$(3.6) \quad k_{\lambda\mu} = 2A_{[\lambda}B_{\mu]}.$$

Furthermore, making use of (3.2) and (3.5), we obtain

$$(3.7) \quad (a) A^\lambda (= h^{\lambda\nu} A_\nu) : (0, 0, 1, 1), \quad (b) B^\lambda (= h^{\lambda\nu} B_\nu) : (e^t, 0, 0, 0),$$

and

$$(3.8) \quad \begin{aligned} (a) A_\alpha A^\alpha = 0, \quad (b) A_\alpha B^\alpha = 0, \quad (c) k_{\lambda\alpha} A^\alpha = 0, \\ (d) \nabla_\lambda A_\mu = 0, \quad (e) \nabla_\omega B_\mu = A_\omega B_\mu. \end{aligned}$$

THEOREM 3.1. *In UFT X_4 , the vector A_λ given by (3.5)(a) is a solution of the condition (2.9). In this case, the corresponding Einstein connection may be given by*

$$(3.9) \quad \Gamma_{\lambda\mu}^\nu = 2A_{[\lambda}B_{\mu]}A^\nu.$$

And its curvature tensor $R_{\lambda\mu\nu}^\omega$ may be given by

$$(3.10) \quad R_{\lambda\mu\nu}^\omega = 2A_\lambda A_{[\mu}B_{\nu]}A^\omega,$$

Proof. Substituting the vector A_λ into the condition (2.9), the vector A_λ is a solution of the condition (2.9) iff, in virtue of (3.8)(c),

$$(3.11) \quad \nabla_\nu k_{\lambda\mu} = -2k_{\nu[\lambda}A_{\mu]}.$$

But, making use of (3.6), (3.8)(d) and (3.8)(e),

$$(3.12) \quad \nabla_\nu k_{\lambda\mu} = A_\lambda A_\nu B_\mu - A_\mu A_\nu B_\lambda = -2k_{\nu[\lambda}A_{\mu]},$$

which implies that the vector A_λ is a solution of the condition (2.9). Substituting the vector A_λ into (2.8), making use (3.6) and (3.8)(c), and remembering $\{\lambda^\nu{}_\mu\} = 0$, we obtain an Einstein connection (3.9). Substituting the Einstein connection (3.9) into (2.6), we obtain (3.10) by a straightforward computation. \square

Conclusion. In virtue of Theorem 3.1, if UFT X_4 is endowed with the basic tensor $g_{\lambda\mu}$ given by (3.1), then an Einstein connection $\Gamma_{\lambda\mu}^\nu$ is given by (3.9), which satisfy (2.5)(a). In the next, since the contracted curvature tensor $R_{\lambda\mu}$ with respect to the connection (3.9) is given by $R_{\lambda\mu} = 0$, in virtue of (3.10), the field equation (2.5)(c) is satisfied automatically. And since the field equation (2.5)(b) is equivalent to $\partial_{[\lambda}P_{\mu]} = 0$, the field equation (2.5)(b) is satisfied by a vector $P_\mu = \partial_\mu P$, that is, the vector $P_\mu = \partial_\mu P$ is an Einstein's vector.

4. An algebraic solution of field equations in UFT X_4

Assume that we have a particular solution $g_{\lambda\mu}$, $\Gamma_{\lambda\mu}^\nu$ of (2.4) and (2.5). The question arises whether there exist a tensor $\bar{g}_{\lambda\mu}$ which together with the $\bar{\Gamma}_{\lambda\mu}^\nu$ is a solution of (2.4) and (2.5). In order to answer this question we put $\bar{g}_{\lambda\mu}$ in the form

$$(4.1) \quad \bar{g}_{\lambda\mu} = g_{\lambda\mu} + X_{\lambda\mu}$$

where the tensor $X_{\lambda\mu}$ has to be founded. From now on, we shall hold to the following agreement : If T is a function of $g_{\lambda\mu}$, then we denote by \bar{T} the same function of $\bar{g}_{\lambda\mu}$. If, in particular, T is a tensor, so is \bar{T} . From (4.1), we obtain

$$(4.2) \quad (a) \bar{h}_{\lambda\mu} = h_{\lambda\mu} + p_{\lambda\mu} \quad (b) \bar{k}_{\lambda\mu} = k_{\lambda\mu} + q_{\lambda\mu},$$

where $p_{\lambda\mu}$ and $q_{\lambda\mu}$ are the symmetric part and the skew-symmetric part of the tensor $X_{\lambda\mu}$, respectively. And we assume that $\det((\bar{h}_{\lambda\mu})) \neq 0$. we may define a unique tensor $\bar{h}^{\lambda\nu}$ ($= \bar{h}^{\nu\lambda}$) by

$$(4.3) \quad \bar{h}_{\lambda\mu} \bar{h}^{\lambda\nu} = \delta_\mu^\nu.$$

THEOREM 4.1. *If we put*

$$(4.4) \quad \overline{\{\lambda^\omega \nu\}} = \{\lambda^\omega \nu\} + P_{\lambda\nu}^\omega,$$

then $P_{\lambda\nu}^\omega$ is a tensor symmetric in the indices λ and ν , and it is given by

$$(4.5) \quad P_{\lambda\mu}^\nu = \frac{1}{2} \bar{h}^{\nu\alpha} (\nabla_\lambda p_{\mu\alpha} + \nabla_\mu p_{\alpha\lambda} - \nabla_\alpha p_{\lambda\mu}).$$

Proof. By the law of transformation of the Christoffel symbols, $P_{\lambda\mu}^\nu = \overline{\{\lambda^\omega \nu\}} - \{\lambda^\omega \nu\}$ is a tensor symmetric in the indices λ and ν . Multiplying by $\bar{h}_{\omega\mu}$ and summing for ω on both sides of (4.4), and using the expression (4.2)(a) in the right-hand member of (4.4), we obtain

$$(4.6) \quad \overline{[\lambda\nu, \mu]} = [\lambda\nu, \mu] + p_{\mu\alpha} \{\lambda^\alpha \nu\} + \bar{h}_{\mu\alpha} P_{\lambda\nu}^\alpha.$$

In accordance with the definition of the Christoffel symbols we have, from (4.2)(a),

$$(4.7) \quad \overline{[\lambda\nu, \mu]} = [\lambda\nu, \mu] + [\lambda\nu, \mu]_p,$$

where $[\lambda\nu, \mu]_p$ are the Christoffel symbols of the first kind formed with respect to $p_{\lambda\nu}$. Substituting (4.7) into (4.6), we obtain

$$(4.8) \quad [\lambda\nu, \mu]_p = p_{\mu\alpha} \{\lambda^\alpha \nu\} + \bar{h}_{\mu\alpha} P_{\lambda\nu}^\alpha.$$

If we add to this equation the one obtained by interchanging λ and μ , then the result may be written

$$(4.9) \quad \nabla_\nu p_{\lambda\mu} = \bar{h}_{\mu\alpha} P_{\lambda\nu}^\alpha + \bar{h}_{\lambda\alpha} P_{\mu\nu}^\alpha.$$

Subtracting this equation from the sum of the two others which are obtained from it by cyclic permutation of the indices λ , μ and ν , we obtain (4.5). \square

In order to obtain an algebraic solution $\bar{g}_{\lambda\mu}$, $\bar{\Gamma}_{\lambda\mu}^\nu$ by means of a particular solution $g_{\lambda\mu}$, $\Gamma_{\lambda\mu}^\nu$ of (2.4) and (2.5), where $g_{\lambda\mu}$ and $\Gamma_{\lambda\mu}^\nu$ are given by (3.1) and (3.8), let us consider a tensor $\bar{g}_{\lambda\mu}$ given by the following matrix :

$$(4.10) \quad ((\bar{g}_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2e^t & 0 & 1 & 0 \\ -2e^t & 0 & 0 & -1 \end{pmatrix},$$

where $t = x^3 - x^4$. The tensor $\bar{g}_{\lambda\mu}$ may be split into its symmetric part $\bar{h}_{\lambda\mu}$ and skew-symmetric part $\bar{k}_{\lambda\mu}$ given by the following matrices :

$$(4.11) \quad ((\bar{h}_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & e^t & -e^t \\ 0 & 1 & 0 & 0 \\ e^t & 0 & 1 & 0 \\ -e^t & 0 & 0 & -1 \end{pmatrix}.$$

$$(4.12) \quad ((\bar{k}_{\lambda\mu})) = \begin{pmatrix} 0 & 0 & -e^t & e^t \\ 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 \\ -e^t & 0 & 0 & 0 \end{pmatrix}.$$

Hence the tensor $\bar{h}_{\lambda\mu}$ may be split into $h_{\lambda\mu}$ and $p_{\lambda\mu}$ given by the following matrices :

$$(4.13) \quad ((h_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$(4.14) \quad ((p_{\lambda\mu})) = \begin{pmatrix} 0 & 0 & e^t & -e^t \\ 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 \\ -e^t & 0 & 0 & 0 \end{pmatrix}.$$

Hence we obtain

$$(4.15) \quad (a) \bar{h}_{\lambda\mu} = h_{\lambda\mu} + p_{\lambda\mu} \quad (b) q_{\lambda\mu} = 0, \quad (c) \bar{k}_{\lambda\mu} = k_{\lambda\mu} (= 2A_{[\lambda}B_{\mu]}).$$

Since

$$(4.16) \quad (a) \det((\bar{g}_{\lambda\mu})) = -1, \quad (b) \det((\bar{h}_{\lambda\mu})) = -1, \quad (c) \det((\bar{k}_{\lambda\mu})) = 0,$$

we can choose the tensor $\bar{g}_{\lambda\mu}$ given by (4.10) as a basic tensor in UFT X_4 , by the assumption (2.2).

THEOREM 4.2. *In UFT X_4 , for the vectors A_λ and B_λ given by (3.5), the following relations hold.*

$$(4.17) \quad \begin{aligned} (a) p_{\lambda\mu} &= 2A_{(\lambda}B_{\mu)} & (b) p_{\lambda\alpha}A^\alpha &= 0, & (c) p_{\lambda\alpha}B^\alpha &= A_\lambda B_\alpha B^\alpha, \\ (d) A_\alpha \bar{h}^{\alpha\nu} &= A^\nu, & (e) B_\alpha \bar{h}^{\alpha\nu} &= B^\nu - A^\nu B_\beta B^\beta \\ (f) P_{\lambda\mu}^\nu &= A_\lambda A_\mu (B^\nu - A^\nu B_\beta B^\beta) \end{aligned}$$

Proof. A simple inspection based on (3.6), (3.8) and (4.15) shows (4.17)(a)~(d). From (4.15) and (4.17)(c), we obtain

$$(4.18) \quad \bar{h}_{\lambda\alpha}B^\alpha = B_\lambda + A_\lambda B_\alpha B^\alpha$$

Multiplying $\bar{h}^{\lambda\beta}$ on both sides of (4.18) and summing for λ , we obtain

$$(4.19) \quad B^\beta = \bar{h}^{\lambda\beta}B_\lambda + A^\beta B_\alpha B^\alpha,$$

which implies (4.17)(e). Next, substituting (4.17)(a) in (4.5), and making use of (3.8) and (4.17)(e), obtain

$$(4.20) \quad P_{\lambda\mu}^\nu = \bar{h}^{\nu\alpha}A_\mu A_\lambda B_\alpha = A_\lambda A_\mu (B^\nu - A^\nu B_\beta B^\beta).$$

□

THEOREM 4.3. *In UFT X_4 , let $g_{\lambda\mu}$ and $\Gamma_{\lambda\mu}^\nu$ be given by (3.1) and (3.9), respectively. For the basic tensor $\bar{g}_{\lambda\mu}$ given by (4.10), let $\bar{\Gamma}_{\lambda\mu}^\nu$ be a connection with the same torsion tensor as $\Gamma_{\lambda\mu}^\nu$. Then $\bar{\Gamma}_{\lambda\mu}^\nu$ is an Einstein connection which is given by*

$$(4.21) \quad \bar{\Gamma}_{\lambda\mu}^\nu = 2A_{[\lambda}B_{\mu]}A^\nu + A_\lambda A_\mu (B^\nu - A^\nu B_\beta B^\beta).$$

And its curvature tensor $\bar{R}_{\lambda\mu\nu}^\omega$ may be given by

$$(4.22) \quad \bar{R}_{\lambda\mu\nu}^\omega = 2A_\lambda A_{[\mu}B_{\nu]}A^\omega.$$

Proof. Since $\bar{\Gamma}_{\lambda\mu}^\nu$ is a connection with the same torsion tensor as $\Gamma_{\lambda\mu}^\nu$,

$$(4.23) \quad \bar{S}_{\lambda\mu}^\nu = k_{\lambda\mu}A^\nu = 2A_{[\lambda}B_{\mu]}A^\nu.$$

Hence, in virtue of Theorem 2.1, and making use of (3.8)(c), (4.4), (4.5) and (4.20), and remembering $\{\lambda^\nu{}_\mu\} = 0$, the connection $\bar{\Gamma}^\nu{}_{\lambda\mu}$ may be given by

$$(4.24) \quad \bar{\Gamma}^\nu{}_{\lambda\mu} = \overline{\{\lambda^\nu{}_\mu\}} + k_{\lambda\mu}A^\nu = A_\lambda A_\mu (B^\nu - A^\nu B_\beta B^\beta) + 2A_{[\lambda} B_{\mu]} A^\nu.$$

And, in virtue of Theorem 2.2, this connection (4.24) is an Einstein connection if and only if, making use of (3.8)(c),

$$(4.25) \quad \bar{\nabla}_\nu k_{\lambda\mu} = -2k_{\nu[\lambda} A_{\mu]}$$

But since, making use of (4.17)(f), (3.6) and (3.8),

$$(4.26) \quad \begin{aligned} \bar{\nabla}_\nu k_{\lambda\mu} &= \partial_\nu k_{\lambda\mu} - k_{\alpha\mu} \overline{\{\lambda^\alpha{}_\nu\}} - k_{\lambda\alpha} \overline{\{\mu^\alpha{}_\nu\}} \\ &= \partial_\nu k_{\lambda\mu} - k_{\alpha\mu} P^\alpha_{\lambda\nu} - k_{\lambda\alpha} P^\alpha_{\mu\nu} \\ &= \partial_\nu k_{\lambda\mu} = \nabla_\nu k_{\lambda\mu} = -2k_{\nu[\lambda} A_{\mu]}, \end{aligned}$$

the connection (4.21) is an Einstein connection which satisfies (2.4). Next Substituting (4.21) into the curvature tensor :

$$(4.27) \quad \bar{R}^\omega{}_{\lambda\mu\nu} = \partial_\mu \bar{\Gamma}^\omega{}_{\lambda\nu} - \partial_\nu \bar{\Gamma}^\omega{}_{\lambda\mu} + \bar{\Gamma}^\alpha{}_{\lambda\nu} \bar{\Gamma}^\omega{}_{\alpha\mu} - \bar{\Gamma}^\alpha{}_{\lambda\mu} \bar{\Gamma}^\omega{}_{\alpha\nu},$$

we obtain (4.22), by a straightforward computation. \square

Conclusion. In virtue of Theorem 4.3, if UFT X_4 is endowed with the basic tensor $\bar{g}_{\lambda\mu}$ given by (4.10), then an Einstein connection $\bar{\Gamma}^\nu{}_{\lambda\mu}$ is given by (4.21), which satisfy (2.5)(a). Furthermore, since from (4.22), the contracted curvature tensor $\bar{R}_{\lambda\mu}$ with respect to the connection (4.21) is given by $\bar{R}_{\lambda\mu} = 0$, the field equation (2.5)(c) is satisfied automatically. On the other hand, since the field equation (2.5)(b) is equivalent to $\partial_{[\lambda} P_{\mu]} = 0$, the field equation (2.5)(b) is satisfied by a vector $P_\mu = \partial_\mu P$, that is, the vector $P_\mu = \partial_\mu P$ is an Einstein's vector. Consequently, for a particular solution $g_{\lambda\mu}$, $\Gamma^\nu{}_{\lambda\mu}$ of (2.4) and (2.5), $\bar{g}_{\lambda\mu}$ is an algebraic solution which together with the $\bar{\Gamma}^\nu{}_{\lambda\mu}$ is a solution of (2.4) and (2.5).

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